

Inverses of symmetric, diagonally dominant positive matrices and applications

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Abstract

We prove tight bounds for the ∞ -norm of the inverse of a symmetric, diagonally dominant positive matrix J ; in particular, we show that $\|J^{-1}\|_\infty$ is uniquely maximized among all such J . We also prove a new lower-bound form of Hadamard's inequality for the determinant of diagonally dominant positive matrices and an improved upper bound for diagonally balanced positive matrices. Applications of our results include numerical stability for linear systems, bounds on inverses of differentiable functions, and consistency of the maximum likelihood equations for random graph distributions.

1 Introduction

An $n \times n$ real matrix J is *diagonally dominant* if

$$\Delta_i(J) := |J_{ii}| - \sum_{j \neq i} |J_{ij}| \geq 0, \quad \text{for } i = 1, \dots, n.$$

Irreducible, diagonally dominant matrices are always invertible, and such matrices arise often in theory and applications. As a recent example, the work of Spielman and Teng [13, 14] shows how to solve diagonally dominant linear systems in nearly-linear time in the input size, a fundamental advance in algorithmic complexity theory and numerical computation. In this paper, we study the mathematical properties of the inverse and determinant of a large class of diagonally dominant matrices.

By far, the most useful information about the inverses of such matrices in applications are bounds on their norms, so we discuss these properties first. A classical result of Varah [16] states that if J is *strictly diagonally dominant*, i.e. if $\Delta_i(J) > 0$ for $1 \leq i \leq n$, then the inverse of J satisfies the bound:

$$\|J^{-1}\|_\infty \leq \max_{1 \leq i \leq n} \frac{1}{\Delta_i(J)}.$$

Here, $\|\cdot\|_\infty$ is the maximum absolute row sum of a matrix, which is the matrix norm induced by the infinity norm $|\cdot|_\infty$ on vectors in \mathbb{R}^n . Generalizations of this basic estimate can be found in [17], [12], and [9], but all involve the quantity $\max_{1 \leq i \leq n} 1/\Delta_i(J)$. In practice, however, one sometimes needs bounds when $\Delta_i(J) = 0$

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for some i , in which case the estimates appearing in [16, 17, 12, 9] do not apply. A particularly interesting case is when $\Delta_i(J) = 0$ for all i ; we call such matrices *diagonally balanced*.

We prove a tight bound on $\|J^{-1}\|_\infty$ for symmetric diagonally dominant J with positive entries that is independent of the quantities $\Delta_i(J)$, and thus also of the maximum entry of J . Let $S = (n-2)I_n + \mathbf{1}_n \mathbf{1}_n^\top$ be the diagonally balanced matrix whose off-diagonal entries are all equal to 1, and recall the *Loewner partial ordering* on symmetric matrices: $A \succeq B$ means that $A - B$ is positive semidefinite. We shall also write $A \geq B$ if $A - B$ is a nonnegative matrix. In Lemma 7.1 below, we show that if $\ell > 0$ and J is a symmetric diagonally dominant matrix satisfying $J \geq \ell S$, then $J \succeq \ell S \succ 0$; in particular, J is invertible. Throughout this paper, I_n and $\mathbf{1}_n$ denote the $n \times n$ identity matrix and the n -dimensional column vector consisting of all ones, respectively. We also write I and $\mathbf{1}$ if the dimension n is understood.

The following is our first main result.

Theorem 1.1. *Let $n \geq 3$. For any symmetric diagonally dominant matrix J with $J_{ij} \geq \ell > 0$, we have*

$$\|J^{-1}\|_\infty \leq \frac{1}{\ell} \|S^{-1}\|_\infty = \frac{3n-4}{2\ell(n-2)(n-1)}.$$

Moreover, equality is achieved if and only if $J = \ell S$.

In other words, the map $J \mapsto \|J^{-1}\|_\infty$ over the (translated) cone of symmetric diagonally dominant matrices $J \geq \ell S$ is maximized uniquely at the point of the cone; i.e., when $J = \ell S$. If the off-diagonal entries of J are bounded above by m and the largest of the diagonal dominances $\Delta_i(J)$ is δ , we also have the trivial lower bound:

$$\frac{1}{2m(n-1) + \delta} \leq \|J^{-1}\|_\infty,$$

which follows from submultiplicativity of the matrix norm $\|\cdot\|_\infty$. Therefore, the value of $\|J^{-1}\|_\infty = \Theta(\frac{1}{n})$ is tightly constrained for bounded ℓ, m and δ .

To probe some of the difficulty of Theorem 1.1, we first derive an estimate from standard bounds in matrix analysis. The relation $J \succeq \ell S$ is equivalent to $J^{-1} \preceq (\ell S)^{-1} = \frac{1}{\ell} S^{-1}$ [5, Corollary 7.7.4], and therefore by a basic inequality [6, p. 214, Ex. 14], we have $\|J^{-1}\| \leq \frac{1}{\ell} \|S^{-1}\|$ for any unitarily invariant matrix norm $\|\cdot\|$, such as the spectral, Frobenius, or Ky-Fan norms. It follows that

$$\|J^{-1}\|_\infty \leq \sqrt{n} \|J^{-1}\|_2 \leq \frac{\sqrt{n}}{\ell} \|S^{-1}\|_2 = \frac{\sqrt{n}}{(n-2)\ell}. \quad (1)$$

However, this bound is $O(\frac{1}{\sqrt{n}})$, whereas the bound given in Theorem 1.1 is $O(\frac{1}{n})$. In some applications this difference can be crucial. For instance, we explain in Section 2 how Theorem 1.1 proves the consistency of the maximum likelihood estimator for some random graph distributions inspired by neuroscience [4].

As the following example shows, Theorem 1.1 fails to hold if we relax the assumption that J be symmetric.

Example 1.2. For $t \geq 0$, consider the following diagonally balanced matrices and their inverses:

$$J_t = \begin{pmatrix} 2+t & 1 & 1+t \\ 1 & 2+t & 1+t \\ 1 & 1 & 2 \end{pmatrix}; \quad J_t^{-1} = \frac{1}{4} \begin{pmatrix} \frac{t+3}{t+1} & \frac{t-1}{t+1} & -t-1 \\ \frac{t-1}{t+1} & \frac{t+3}{t+1} & -t-1 \\ -1 & -1 & t+3 \end{pmatrix}.$$

Since some entries of J_t^{-1} diverge, we have $\|J_t^{-1}\|_\infty \rightarrow \infty$ as $t \rightarrow \infty$. □

One standard approach to proving norm estimates such as the one in Theorem 1.1 is a perturbation analysis. More specifically, given a symmetric diagonally dominant J with entries bounded below by ℓ , one tries to replace each entry J_{ij} by ℓ and prove that the norm of the inverse of the resulting matrix is larger. However, such a method will not succeed, even in the balanced case, as the following examples demonstrate.

Example 1.3. Consider the following two balanced matrices:

$$J = \begin{pmatrix} 12 & 4 & 1 & 7 \\ 4 & 9 & 3 & 2 \\ 1 & 3 & 7 & 3 \\ 7 & 2 & 3 & 12 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 9 & 1 & 1 & 7 \\ 1 & 6 & 3 & 2 \\ 1 & 3 & 7 & 3 \\ 7 & 2 & 3 & 12 \end{pmatrix}.$$

Here, H is obtained from J by changing its $(1, 2)$ -entry to 1 and then keeping the resulting matrix balanced. A direct computation shows that $\|H^{-1}\|_{\infty} < \|J^{-1}\|_{\infty}$. As another example, the following two matrices:

$$J = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

also have $\|H^{-1}\|_{\infty} < \|J^{-1}\|_{\infty}$. Here, the $(1, 2)$ -entry was changed *without* keeping the matrix balanced. \square

As it will be an essential calculation for our proof of Theorem 1.1, we study (in Section 4) the limit

$$N = \lim_{t \rightarrow \infty} (S + tP)^{-1}, \quad (2)$$

where P is a symmetric diagonally dominant matrix with $P_{ij} \in \{0, 1\}$ and $\Delta_i(P) \in \{0, 2\}$. Each such matrix P is a *signless Laplacian* of an undirected, unweighted graph G , possibly with self-loops. We compute the limit (2) in terms of the bipartition structure of the connected components of G . In particular, if G is connected and not bipartite, this limit N is the zero matrix (see Corollary 4.5). For some recent work on the general eigenstructure of signless Laplacians, we refer the reader to [2] and the references therein.

The following example reveals some of the combinatorics at work underlying Theorem 1.1.

Example 1.4. Consider the chain graph G with edges $\{1, n\}$ and $\{i, i+1\}$ for $i = 1, \dots, n-1$. If n is odd then $N = 0$ since G is not bipartite, while if n is even, the limit N has alternating entries:

$$N_{ij} = \frac{(-1)^{i+j}}{n(n-2)}.$$

As another example, consider the star graph G , which has edges $\{1, i\}$ for $i = 2, \dots, n$. In this case,

$$N_{1i} = N_{i1} = -\frac{1}{2(n-1)(n-2)}, \text{ for } i = 2, \dots, n; \quad \text{and} \quad N_{ij} = \frac{1}{2(n-1)(n-2)}, \text{ otherwise.} \quad \square$$

We next describe our second collection of main results concerning the determinant and adjugate of positive, diagonally dominant symmetric matrices J . Recall the well-known formula:

$$J^{-1} = \frac{J^*}{\det(J)},$$

in which J^* is the adjugate (or classical adjoint) of J . To prove Theorem 1.1, one might first try to bound $\|J^*\|_{\infty}$ from above and $\det(J)$ from below, preferably separately. We first focus on estimating the latter, which is already nontrivial.

It is classical that the determinant of a positive semidefinite matrix A is bounded above by the product of its diagonal entries:

$$0 \leq \det(A) \leq \prod_{i=1}^n A_{ii}.$$

This well-known result is sometimes called Hadamard's inequality [5, Theorem 7.8.1]. A lower bound of this form, however, is not possible without additional assumptions. Surprisingly, there is such an inequality when J is diagonally dominant with positive entries; and, when J is also balanced, an improved upper bound.

Theorem 1.5. *Let $n \geq 3$, and let J be an $n \times n$ symmetric matrix with off-diagonal entries $m \geq J_{ij} \geq \ell > 0$.*

(a) *If J is diagonally dominant, then*

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} \geq \left(1 - \frac{1}{2(n-2)} \sqrt{\frac{m}{\ell}} \left(1 + \frac{m}{\ell}\right)\right)^{n-1} \rightarrow \exp\left(-\frac{1}{2} \sqrt{\frac{m}{\ell}} \left(1 + \frac{m}{\ell}\right)\right) \quad \text{as } n \rightarrow \infty.$$

(b) *If J is diagonally balanced, then*

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} \leq \exp\left(-\frac{\ell^2}{4m^2}\right).$$

The bounds above depend on the largest off-diagonal entry of J (in an essential way for (a); see Example 7.5), and thus are ill-adapted to prove Theorem 1.1. For instance, combining Theorem 1.5 (a) with Hadamard's inequality applied to the positive definite J^* in the obvious way gives estimates which are worse than (1). Nevertheless, Theorem 1.5 should be of independent interest, and we prove it in Section 7.

As an immediate application of Theorems 1.1 and 1.5, we obtain a bound on J^* for balanced, positive J .

Corollary 1.6. *Let $n \geq 3$, and let J be a symmetric $n \times n$ matrix J with off-diagonal entries $m \geq J_{ij} \geq \ell > 0$. If J is diagonally balanced, then the adjugate J^* satisfies the bound:*

$$\frac{\|J^*\|_\infty}{\prod_{i=1}^n J_{ii}} \leq \frac{3n-4}{2\ell(n-2)(n-1)} e^{-\ell^2/(4m^2)}.$$

Remark 1.7. We do not know how to prove this result other than by appealing to Theorems 1.1 and 1.5.

We finish this introduction with a brief overview of how our main results are proved. Theorem 1.1 will be generalized in Theorem 6.1 where we consider diagonally dominant matrices $J \geq S(\alpha, \ell) := \alpha I_n + \ell \mathbf{1}_n \mathbf{1}_n^\top$ with $\alpha \geq (n-2)\ell > 0$. We break up the proof of this general theorem into three main steps in Sections 3 to 6, where we write $S(\alpha, \ell)$ as S for simplicity. The first step considers the problem of maximizing $\|J^{-1}\|_\infty$ over symmetric diagonally dominant J with J_{ij} and $\Delta_i(J)$ in some finite intervals (Section 3). In this case, the maximum is achieved when J is on the *corners* of the space; namely, when J_{ij} and $\Delta_i(J)$ are one of the endpoints of the finite intervals. In the second step (Section 4), we analyze the behavior of corner matrices at infinity, showing that the limit of $\|(S + tP)^{-1}\|_\infty$ as $t \rightarrow \infty$ when P is a signless Laplacian ($P_{ij} \in \{0, 1\}$ and $\Delta_i(P) \in \{0, 2\}$) is at most $\|S^{-1}\|_\infty$. Combined with the first step, this shows that the matrix S maximizes $\|J^{-1}\|_\infty$ over the space of symmetric diagonally dominant matrices $J \geq S$. The remainder of the argument deals with the behavior of $\|J^{-1}\|_\infty$ near S to show that S is indeed the unique maximizer (Section 5). All three steps are combined in Section 6. The inequalities of Theorem 1.5 are proved using a block matrix factorization in Section 7. We conclude with a brief discussion of open questions in Section 8.

2 Applications

The (∞ -norm) condition number $\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty$ of a matrix A plays an important role in numerical linear algebra. For instance, the relative error $|x - \hat{x}|_\infty / |x|_\infty$ of an approximate solution \hat{x} to a set of linear equations $Ax = b$ is bounded by the product of $\kappa_\infty(A)$ and the relative size of the residual, $|b - A\hat{x}|_\infty / |b|_\infty$ (e.g., [5, p. 338]). Directly from Theorem 1.1, we may bound the condition number of a positive, diagonally dominant symmetric matrix. Thus, numerical linear computation involving such matrices is well-behaved.

Corollary 2.1. *The condition number $\kappa_\infty(A)$ of a positive, diagonally dominant symmetric $n \times n$ matrix A with largest off-diagonal entry m , smallest entry ℓ , and largest diagonal dominance δ satisfies:*

$$\kappa_\infty(A) \leq \frac{(2m(n-1) + \delta)(3n-4)}{2\ell(n-2)(n-1)}.$$

We next discuss another application of Theorem 1.1 to the numerical stability of inverses of a large family of functions. Let U be a convex open subset of \mathbb{R}^n , $n \geq 3$, and consider the map $F: U \rightarrow \mathbb{R}^n$ given by

$$F(x) = (F_1(x), \dots, F_n(x)), \quad F_i(x) = \sum_{j=1}^n g_{ij}(x_i + x_j), \quad (3)$$

in which $g_{ij}: \mathbb{R} \rightarrow \mathbb{R}$ are any continuously differentiable functions with $g_{ij} = g_{ji}$, $g'_{ij}(x_i + x_j) \geq \ell > 0$, and $g'_{ii}(2x_i) \geq 0$ for all $1 \leq i, j \leq n$ and $x \in U$.

Theorem 2.2. *For any $\theta, \hat{\theta} \in F(U)$, we have:*

$$|F^{-1}(\theta) - F^{-1}(\hat{\theta})|_\infty \leq \frac{3n-4}{2\ell(n-1)(n-2)} |\theta - \hat{\theta}|_\infty.$$

Proof. The function F is continuously differentiable on U , and its Jacobian $J = J_F(x)$ satisfies:

$$J_{ij}(x) = \frac{\partial F_i(x)}{\partial x_j} = g'_{ij}(x_i + x_j) \geq \ell, \quad J_{ii}(x) = \frac{\partial F_i(x)}{\partial x_i} = 2g'_{ii}(2x_i) + \sum_{j \neq i} g'_{ij}(x_i + x_j) \geq \sum_{j \neq i} J_{ij}(x).$$

In particular, J is symmetric and diagonally dominant with off-diagonal entries bounded below by $\ell > 0$.

By the mean-value theorem for vector-valued functions [8, p. 341], for each pair $x, y \in U$, we can write:

$$F(x) - F(y) = \tilde{J}(x - y), \quad (4)$$

in which the matrix $\tilde{J} = \int_0^1 J(tx + (1-t)y) dt$ is the element-wise average of the Jacobians on the line segment between x and y . Thus, \tilde{J} is also symmetric and diagonally dominant with $\tilde{J}_{ij} \geq \ell$. In particular, \tilde{J} is invertible, which shows that the map F is invertible on U . By substituting $\theta = F(x)$ and $\hat{\theta} = F(y)$ into (4), inverting \tilde{J} , and applying Theorem 1.1 to \tilde{J}^{-1} , we arrive at the inequality stated in the theorem. \square

We close this section by explaining an application of Theorem 2.2 to probability and statistics. In [4], the maximum entropy distribution on (undirected) weighted graphs $A = (A_{ij})_{i,j=1}^n$ given an expected vertex degree sequence $d = (d_1, \dots, d_n) \in \mathbb{R}^n$ is studied, extending the work of Chatterjee, Diaconis, and Sly [1] in the case of unweighted graphs. When the graphs have edges in $[0, \infty)$, this distribution \mathbb{P}_θ has random weights A_{ij} that are independent exponential variables with $\mathbb{E}[A_{ij}] = 1/(\theta_i + \theta_j) > 0$, where the parameters $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ are chosen so that the expected degree sequence of the distribution is equal to d .

Given θ with $1/m \leq (\theta_i + \theta_j)^2 \leq 1/\ell$ for $i \neq j$, suppose we draw a sample graph $\hat{G} \sim \mathbb{P}_\theta$ with weights $\hat{A} = (\hat{A}_{ij})$, and let $\hat{d}_i = \sum_{j \neq i} \hat{A}_{ij}$ be the degree sequence of \hat{G} . The maximum likelihood estimator $\hat{\theta}$ for θ solves the moment-matching equations (the *retina equations* in theoretical neuroscience [4]):

$$\hat{d}_i = \sum_{j \neq i} \frac{1}{\hat{\theta}_i + \hat{\theta}_j}, \quad \text{for } i = 1, \dots, n. \quad (5)$$

It can be shown that a solution $\hat{\theta}$ to (5) with $\hat{\theta}_i + \hat{\theta}_j > 0$, $i \neq j$, is unique almost surely and, more surprisingly, that the estimator $\hat{\theta}$ is *consistent*. In other words, the degree sequence of one sample from a maximum entropy distribution essentially determines the distribution for large n . We remark that solutions $\hat{\theta}$ to equations (5) also exhibit a rich combinatorial structure which has been explored (within a more general framework) by Sanyal, Sturmfels, and Vinzant using matroid theory and algebraic geometry [11].

To see how Theorem 2.2 applies to this context, consider the function F determined as in (3) by setting $g_{ij}(z) = -1/z$ for $i \neq j$ and $g_{ii} = 0$ otherwise. In this case, equations (5) are concisely expressed as $\hat{d} = -F(\hat{\theta})$. Using Theorem 2.2 and arguments from large deviation theory, one can show that given any $k > 1$ and for sufficiently large n , we have the estimate:

$$|\theta - \hat{\theta}|_\infty \leq \frac{150\sqrt{m}}{\ell} \sqrt{\frac{k \log n}{n}} \quad \text{with probability } \geq 1 - \frac{3}{n^{k-1}}. \quad (6)$$

Thus, $|\theta - \hat{\theta}|_\infty \rightarrow 0$ with high probability as $n \rightarrow \infty$. We note that analogous arguments with the $O(\frac{1}{\sqrt{n}})$ bound from (1) fail to show consistency. We refer the reader to [4] for more details.

3 Reduction to Corners

To prove Theorem 1.1, we need to show that the maximum of $\|J^{-1}\|_\infty$ over the space of symmetric diagonally dominant matrices $J \geq S := \alpha I_n + \ell \mathbf{1}_n \mathbf{1}_n^\top$ is achieved at $J = S$. A priori, it is not even clear that a maximum exists since this space is not compact. In this section we consider maximizing $\|J^{-1}\|_\infty$ over compact sets of symmetric diagonally dominant matrices J , and we show that the maxima occur at the corners of the space. In subsequent sections we analyze these corner matrices in more detail.

Fix $m \geq 1$, and let $\mathcal{D} = \mathcal{D}_m$ denote the set of $n \times n$ matrices of the form $J = S + (m - \ell)P$ where P is some symmetric diagonally dominant matrix satisfying

$$0 \leq P_{ij} \leq 1 \text{ for } i \neq j \quad \text{and} \quad 0 \leq \Delta_i(P) \leq 2 \text{ for } i = 1, \dots, n.$$

We say that $J \in \mathcal{D}$ is a *corner matrix* if

$$P_{ij} \in \{0, 1\} \text{ for } i \neq j \quad \text{and} \quad \Delta_i(P) \in \{0, 2\} \text{ for } i = 1, \dots, n.$$

Equivalently, J is a corner matrix if P is a signless Laplacian matrix. Let \mathcal{T} denote the set of matrices $J \in \mathcal{D}$ that maximize $\|J^{-1}\|_\infty$. This set is closed and nonempty since the function $J \mapsto \|J^{-1}\|_\infty$ is continuous and \mathcal{D} is compact in the usual topology. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the standard column basis for \mathbb{R}^n , and set $\mathbf{e}_{ii} = \mathbf{e}_i \mathbf{e}_i^\top$ and $\mathbf{e}_{ij} = (\mathbf{e}_i + \mathbf{e}_j)(\mathbf{e}_i + \mathbf{e}_j)^\top$ for all $i \neq j$. Our main result in this section is the following.

Proposition 3.1. *Every $J \in \mathcal{T}$ is path-connected to a corner matrix.*

Proof. Let $J \in \mathcal{T}$. We will show that if $\ell < J_{ij} < m$ for some $i \neq j$, there is a path in \mathcal{T} from J to a matrix J' that differs from J only in the (i, j) -entry, with $J'_{ij} \in \{\ell, m\}$. Similarly, if $0 < \Delta_i(J) < 2(m - \ell)$ for some $1 \leq i \leq n$, then we find a suitable J' with $\Delta_i(J') \in \{0, 2(m - \ell)\}$ that differs from J in the (i, i) -entry. Repeatedly applying these steps, it follows that there is a path in \mathcal{T} from any $J \in \mathcal{T}$ to a corner matrix.

For the first part, suppose that $\ell < J_{ij} < m$ for some $i \neq j$. Consider the nonempty, closed set

$$\mathcal{W} = \{J + t\mathbf{e}_{ij} : \ell \leq J_{ij} + t \leq m\} \cap \mathcal{T}.$$

We claim that \mathcal{W} contains a matrix J' with $J'_{ij} \in \{\ell, m\}$. Suppose not, and let $J' \in \mathcal{W}$ be a matrix with minimum (i, j) -entry. By Proposition 3.2 (a) below, $J' + t\mathbf{e}_{ij} \in \mathcal{T}$ for all t in a small neighborhood of the origin. Thus, there is another matrix in \mathcal{T} that has a smaller (i, j) -entry than J' , a contradiction. The proof for the other part is similar (using Proposition 3.2 (b)). \square

To complete the proof of Proposition 3.1, it remains to show the following.

Lemma 3.2. *Let $J \in \mathcal{T}$ and $i \neq j$ be distinct indices in $\{1, \dots, n\}$.*

- (a) *If $1 < J_{ij} < m$, then $J + t\mathbf{e}_{ij} \in \mathcal{T}$ for all $t \in \mathbb{R}$ in some neighborhood of the origin.*
- (b) *If $0 < \Delta_i(J) < 2(m - 1)$, then $J + t\mathbf{e}_{ii} \in \mathcal{T}$ for all $t \in \mathbb{R}$ in some neighborhood of the origin.*

Proof. We only prove (a) as (b) is analogous. Suppose that $1 < J_{ij} < m$ for some $i \neq j$. Let $K = J^{-1}$ and set K_1, \dots, K_n to be its columns. Also, let $q \in \arg \max_{1 \leq p \leq n} |K_p|_1$ and $\mathcal{I} = \{1 \leq p \leq n : K_{pq} \neq 0\}$ so that

$$\sum_{p \in \mathcal{I}} |K_{pq}| = |K_q|_1 = \|K\|_\infty = \|J^{-1}\|_\infty.$$

By the Sherman-Morrison-Woodbury formula, we have

$$(J + t\mathbf{e}_{ij})^{-1} = K - \frac{t}{1 + t(\mathbf{e}_i + \mathbf{e}_j)^\top K(\mathbf{e}_i + \mathbf{e}_j)} K\mathbf{e}_{ij}K = K - \frac{t}{1 + \eta_{ij}t} (K_i + K_j)(K_i + K_j)^\top$$

where $\eta_{ij} := (\mathbf{e}_i + \mathbf{e}_j)^\top K(\mathbf{e}_i + \mathbf{e}_j) > 0$ since $K \succ 0$. The formula implies that for sufficiently small $\varepsilon > 0$ and all $|t| \leq \varepsilon$, the (p, q) -entry of $(J + t\mathbf{e}_{ij})^{-1}$ has the same sign as $K_{pq} \neq 0$ for all $p \in \mathcal{I}$. Let us further suppose that ε is small enough so that $J + t\mathbf{e}_{ij} \in \mathcal{D}$ and $1 + \eta_{ij}t > 0$ for all $|t| \leq \varepsilon$. The 1-norm of the q -th column $(J + t\mathbf{e}_{ij})_q^{-1}$ of the matrix of $(J + t\mathbf{e}_{ij})^{-1}$ now satisfies:

$$|(J + t\mathbf{e}_{ij})_q^{-1}|_1 = \|J^{-1}\|_\infty - \frac{\phi_{ij}t}{1 + \eta_{ij}t} + \frac{\psi_{ij}|t|}{1 + \eta_{ij}t}, \quad \text{for all } |t| \leq \varepsilon$$

where $\phi_{ij} := (K_{iq} + K_{jq}) \sum_{p \in \mathcal{I}} \text{sign}(K_{pq})(K_{ip} + K_{jp})$ and $\psi_{ij} := |K_{iq} + K_{jq}| \sum_{p \notin \mathcal{I}} |K_{ip} + K_{jp}|$. Recalling that $J \in \mathcal{T}$ achieves the maximum $\|J^{-1}\|_\infty$, it follows that

$$\phi_{ij}t \geq \psi_{ij}|t|, \quad \text{for all } |t| < \varepsilon.$$

This inequality implies that $\psi_{ij} = \phi_{ij} = 0$, so $\|(J + t\mathbf{e}_{ij})^{-1}\|_\infty = \|J^{-1}\|_\infty$ for all $|t| < \varepsilon$ as required. \square

4 Corner Combinatorics and Exact Limit Formulae

In the previous section, we saw that corner matrices played an important role in the optimization of $\|J^{-1}\|_\infty$ over all matrices $J \geq S := \alpha I_n + \ell \mathbf{1}_n \mathbf{1}_n^\top$. A corner matrix may be expressed as:

$$J = S + (m - \ell)P,$$

in which P is a symmetric diagonally dominant matrix with $P_{ij} \in \{0, 1\}$ for all $i \neq j$ and $\Delta_i(P) \in \{0, 2\}$ for $1 \leq i \leq n$. Every such matrix P is a *signless Laplacian* of an undirected unweighted graph (possibly with self-loops) $G = (V, E)$ on vertices $V = \{1, \dots, n\}$ and edges E . That is, we can write $P = D + A$ where D is the diagonal degree matrix of G and A is its adjacency matrix. We study the limit (2) using the combinatorics of the graphs associated to the matrix P .

Example 4.1. Let $S = (n - 2)I_n + \mathbf{1}_n \mathbf{1}_n^\top$ and G be the chain graph from Example 1.4. For $n = 4$,

$$S + tP = \begin{pmatrix} 3 + 2t & 1 + t & 1 & 1 + t \\ 1 + t & 3 + 2t & 1 + t & 1 \\ 1 & 1 + t & 3 + 2t & 1 + t \\ 1 + t & 1 & 1 + t & 3 + 2t \end{pmatrix},$$

with inverse:

$$(S + tP)^{-1} = \frac{1}{4(2t + 3)(t + 1)} \begin{pmatrix} t^2 + 5t + 5 & -(t + 1)^2 & t^2 + t - 1 & -(t + 1)^2 \\ -(t + 1)^2 & t^2 + 5t + 5 & -(t + 1)^2 & t^2 + t - 1 \\ t^2 + t - 1 & -(t + 1)^2 & t^2 + 5t + 5 & -(t + 1)^2 \\ -(t + 1)^2 & t^2 + t - 1 & -(t + 1)^2 & t^2 + 5t + 5 \end{pmatrix}.$$

Each entry of $(S + tP)^{-1}$ is a rational function of t , with numerator and denominator both quadratic. Thus, each entry converges to a constant as $t \rightarrow \infty$, and from the expression above, we see that the limit matrix N has entries $N_{ij} = (-1)^{i+j}/8$, as predicted by the formula in Example 1.4.

If one adds the edge $\{1, 3\}$, the corresponding matrix $(S + tP)^{-1}$ is

$$(S + tP)^{-1} = \frac{1}{4(t + 3)(t + 1)} \begin{pmatrix} 2t + 5 & -t - 1 & -1 & -t - 1 \\ -t - 1 & 3t + 5 & -t - 1 & t - 1 \\ -1 & -t - 1 & 2t + 5 & -t - 1 \\ -t - 1 & t - 1 & -t - 1 & 3t + 5 \end{pmatrix}.$$

Thus $N = 0$, as the graph is no longer bipartite (see Corollary 4.5 below). \square

We begin with the following simple fact that allows us to invert certain classes of matrices explicitly.

Lemma 4.2. *The following identity holds for any $\alpha \neq 0$ and $\ell \neq -\alpha/n$:*

$$(\alpha I_n + \ell \mathbf{1}_n \mathbf{1}_n^\top)^{-1} = \frac{1}{\alpha} I_n - \frac{\ell}{\alpha(\alpha + \ell n)} \mathbf{1}_n \mathbf{1}_n^\top.$$

Given a signless Laplacian P , let $L \in \mathbb{R}^{n \times |E|}$ be the incidence matrix of the graph G associated to P ; that is, for every vertex $v \in V$ and edge $e \in E$, we have:

$$L_{v,e} = \begin{cases} 1 & \text{if } v \text{ is in } e \text{ and } e \text{ is not a self-loop,} \\ \sqrt{2} & \text{if } v \text{ is in } e \text{ and } e \text{ is the self-loop } (v, v), \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $P = LL^\top$. Using this decomposition of P , we derive the following formula for N .

Proposition 4.3. *The limit N in (2) satisfies:*

$$N = S^{-1} - S^{-1/2} (X^\top)^\dagger X^\top S^{-1/2},$$

where $X = S^{-1/2}L$ and $(X^\top)^\dagger$ is the Moore-Penrose pseudoinverse of X^\top . Furthermore, $NL = 0$.

Proof. Using the Sherman-Morrison-Woodbury matrix formula to expand $(S + tLL^\top)^{-1}$, we calculate:

$$\begin{aligned} N &= \lim_{t \rightarrow \infty} (S + tLL^\top)^{-1} \\ &= \lim_{t \rightarrow \infty} [S^{-1} - S^{-1}L(t^{-1}I + L^\top S^{-1}L)^{-1}L^\top S^{-1}] \\ &= S^{-1} - S^{-1/2} [\lim_{t \rightarrow \infty} X(t^{-1}I + X^\top X)^{-1}] X^\top S^{-1/2} \\ &= S^{-1} - S^{-1/2} (X^\top)^\dagger X^\top S^{-1/2}, \end{aligned}$$

where in the last step we used an elementary pseudoinverse identity in matrix analysis [5, p. 422, Ex. 9]. To show that $NL = 0$, we note that N is symmetric and that

$$(NL)^\top = L^\top N = X^\top S^{-1/2} - X^\top (X^\top)^\dagger X^\top S^{-1/2} = 0. \quad \square$$

Let N_1, \dots, N_n denote the columns of the matrix N . The following is immediate from $NL = 0$.

Corollary 4.4. *For each edge $\{i, j\}$ of G , we have $N_i = -N_j$. In particular, $N_i = 0$ for each self-loop $\{i, i\}$.*

Suppose that the graph G has connected components G_1, \dots, G_k . After relabeling the vertices, both P and L are block-diagonal with matrices P_1, \dots, P_k and L_1, \dots, L_k along the diagonal. Furthermore, $P_i = L_i L_i^\top$ for each $1 \leq i \leq k$. The components of G also induce a block-structure on the limit N , and we denote these blocks by $N[i, j]$ for $i, j \in \{1, \dots, k\}$. These blocks display many interesting symmetries. Firstly, the entries in each block are all equal up to sign. Secondly, the signs in a block $N[i, j]$ depend on the bipartite structures of the components G_i and G_j . We say that a bipartite graph is (p, q) -bipartite if the partitions are of sizes p and q , respectively. Note that bipartite graphs cannot have self-loops.

Corollary 4.5. *In each block $N[i, j]$, the entries are equal up to sign. If the i -th component G_i of G is not bipartite, then $N[i, j] = N[j, i] = 0$ for all $j = 1, \dots, k$. In particular, $N = 0$ if and only if all components G_i are not bipartite. Suppose G_i is (p_i, q_i) -bipartite and G_j is (p_j, q_j) -bipartite. Then, after relabeling the vertices, the matrix $N[i, j]$ has the block structure*

$$N[i, j] = c_{ij} \begin{pmatrix} \mathbf{1}_{p_i} \mathbf{1}_{p_j}^\top & -\mathbf{1}_{p_i} \mathbf{1}_{q_j}^\top \\ -\mathbf{1}_{q_i} \mathbf{1}_{p_j}^\top & \mathbf{1}_{q_i} \mathbf{1}_{q_j}^\top \end{pmatrix}, \text{ for some constant } c_{ij} \in \mathbb{R}.$$

Now that we understand the block structure of N , we want to compute the constants c_{ij} . Our approach is to simplify the formula in Proposition 4.3 by expressing the incidence matrix L in a more suitable form.

Proposition 4.6. *Let G be a connected graph with n vertices. If G is not bipartite, then $\text{rank } L = n$.*

Proof. It suffices to prove that the positive semidefinite matrix $P = LL^\top$ has rank n ; i.e., if $x \in \mathbb{R}^n$ satisfies $x^\top Px = 0$, then $x = 0$. Write

$$x^\top Px = \sum_{i=1}^n \sum_{j=1}^n P_{ij} x_i x_j = \sum_{\substack{i \neq j \\ \{i,j\} \in E}} (x_i + x_j)^2 + \sum_{i: \{i,i\} \in E} 2x_i^2$$

so that $x^\top Px = 0$ implies

$$x_i = 0 \text{ if } \{i,i\} \in E \quad \text{and} \quad x_i + x_j = 0 \text{ if } \{i,j\} \in E. \quad (7)$$

If G has a self-loop $\{i,i\}$, then $x_i = 0$ by (7). If G has no self-loops, there is an odd cycle $(i_1, i_2, \dots, i_{2m+1})$ for some $m \geq 1$ (because G is not bipartite). Applying condition (7) to each successive edge in this cycle shows $x_{i_1} = \dots = x_{i_{2m+1}} = 0$. In either case $x_i = 0$ for some vertex $i \in V$. A repeated application of (7) then reveals that $x_j = 0$ for all vertices j connected to i . Since G is connected, we have $x = 0$ as desired. \square

Recall that $e_i \in \mathbb{R}^n$ denotes the i -th standard basis column vector.

Proposition 4.7. *Let G be a connected bipartite graph on n vertices. Then $\text{rank } L = n - 1$. Let U be the $n \times (n - 1)$ matrix with columns $e_1 + \sigma_i e_i$ for $2 \leq i \leq n$, where $\sigma_i = -1$ if vertex i is in the same partition as vertex 1 and $\sigma_i = 1$ otherwise. Then, $L = UB$ for some $(n - 1) \times |E|$ matrix B of rank $n - 1$.*

Proof. Recall that the columns of L are of the form $e_i + e_j$ where $\{i, j\} \in E$ (there are no self-loops because G is bipartite). There is a path $(1, i_1, i_2, \dots, i_m, j)$ from vertex 1 to vertex j for each $2 \leq j \leq n$, so

$$e_1 + (-1)^m e_j = (e_1 + e_{i_1}) - (e_{i_1} + e_{i_2}) + \dots + (-1)^m (e_{i_m} + e_j).$$

Thus, $\text{rank } L \geq n - 1$. Conversely, for each edge $\{i, j\}$ where vertex i is in the same partition as vertex 1,

$$e_i + e_j = -(e_1 - e_i) + (e_1 + e_j),$$

so $\text{rank } L \leq n - 1$. This equation also allows us to write $L = UB$ for some matrix B , and the rank condition on B follows from that of L . \square

Recall that G has k components G_1, \dots, G_k and L is block-diagonal with matrices L_1, \dots, L_k . If G_i is bipartite, we write $L_i = U_i B_i$ as in Proposition 4.7. If G_i is not bipartite, we write $L_i = U_i B_i$ where $U_i = I$ is the identity matrix and $B_i = L_i$. Let r be the number of components of G which are bipartite. If U and B are block-diagonal matrices constructed from U_1, \dots, U_k and B_1, \dots, B_k , then $L = UB$ where $U \in \mathbb{R}^{n \times (n-r)}$ and $B \in \mathbb{R}^{(n-r) \times |E|}$ both have rank $(n - r)$. Note that U contains information about the sizes of the bipartitions of each component whereas B contains information about the edges. Let $U_{(p,q)}$ denote the matrix

$$U_{(p,q)} := \begin{pmatrix} \mathbf{1}_{p-1}^\top & \mathbf{1}_q^\top \\ -I_{p-1} & 0 \\ 0 & I_q \end{pmatrix}.$$

After relabeling the vertices, we have

$$U = \begin{pmatrix} U_{(p_1, q_1)} & 0 & \dots & 0 & 0 \\ 0 & U_{(p_2, q_2)} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & U_{(p_r, q_r)} & 0 \\ 0 & 0 & \dots & 0 & I_s \end{pmatrix} = \begin{pmatrix} \tilde{U} & 0 \\ 0 & I_s \end{pmatrix}, \quad (8)$$

where $s = n - \sum_{i=1}^r (p_i + q_i)$ is the total number of vertices in the non-bipartite components of G . Our next result shows that dependence on the matrix B can be removed in Proposition 4.3. This new formula for N also gives us a method to compute its entries explicitly.

Proposition 4.8. *Let U be as in (8). The limit N in (2) satisfies:*

$$N = S^{-1} - S^{-1}U(U^\top S^{-1}U)^{-1}U^\top S^{-1}.$$

Therefore, N depends only on the sizes of the bipartitions of each component G_i .

Proof. Write $L = UB$ and $P = U(BB^\top U^\top)$. First, note that BB^\top is positive definite since $B \in \mathbb{R}^{(n-r) \times |E|}$ has rank $(n-r)$. Now, $U^\top S^{-1}U$ is positive semidefinite, being a congruence of a positive definite matrix S^{-1} (Sylvester's Law of Inertia). Moreover, since U is injective as a linear map and $S^{-1} \succ 0$, for any $x \in \mathbb{R}^{n-r}$:

$$x^\top U^\top S^{-1}Ux = 0 \quad \Rightarrow \quad Ux = 0 \quad \Rightarrow \quad x = 0.$$

Thus, $U^\top S^{-1}U$ is positive definite. Then the eigenvalues of $BB^\top U^\top S^{-1}U$ are all positive, being a product of two positive definite matrices (see [7, Lemma 2]). Therefore, the matrix $t^{-1}I + BB^\top U^\top S^{-1}U$ is invertible for all $t > 0$, and so by the Sherman-Morrison-Woodbury formula, we have

$$\begin{aligned} (S + tP)^{-1} &= (S + tU(BB^\top U^\top))^{-1} \\ &= S^{-1} - S^{-1}U(t^{-1}I + BB^\top U^\top S^{-1}U)^{-1}BB^\top U^\top S^{-1}. \end{aligned}$$

Taking limits of this equation as $t \rightarrow \infty$, the result follows. \square

To state our ultimate formula for N , let us first define:

$$\begin{aligned} \gamma &= \sum_{i=1}^r \frac{(p_i - q_i)^2}{p_i + q_i}, \\ y^\top &= \left(\frac{p_1 - q_1}{p_1 + q_1} (\mathbf{1}_{p_1}^\top, -\mathbf{1}_{q_1}^\top), \dots, \frac{p_r - q_r}{p_r + q_r} (\mathbf{1}_{p_r}^\top, -\mathbf{1}_{q_r}^\top), 0 \cdot \mathbf{1}_s^\top \right), \\ Y &= \begin{pmatrix} \frac{1}{p_1 + q_1} \begin{pmatrix} \mathbf{1}_{p_1} \mathbf{1}_{p_1}^\top & -\mathbf{1}_{p_1} \mathbf{1}_{q_1}^\top \\ -\mathbf{1}_{q_1} \mathbf{1}_{p_1}^\top & \mathbf{1}_{q_1} \mathbf{1}_{q_1}^\top \end{pmatrix} & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & \frac{1}{p_r + q_r} \begin{pmatrix} \mathbf{1}_{p_r} \mathbf{1}_{p_r}^\top & -\mathbf{1}_{p_r} \mathbf{1}_{q_r}^\top \\ -\mathbf{1}_{q_r} \mathbf{1}_{p_r}^\top & \mathbf{1}_{q_r} \mathbf{1}_{q_r}^\top \end{pmatrix} & 0 \\ 0 & \dots & 0 & 0 \cdot I_s \end{pmatrix}. \end{aligned}$$

Proposition 4.9. *Set γ , y , and Y as above, which depend only on the bipartite structures of the components of the underlying graph G . We have the following formula for the limit in (2):*

$$N = \lim_{t \rightarrow \infty} (S + tP)^{-1} = \frac{1}{\alpha} Y - \frac{\ell}{\alpha(\alpha + \ell\gamma)} yy^\top.$$

Proof. We outline the computation of N . For simplicity, let us write $S^{-1} = aI_n - b\mathbf{1}_n \mathbf{1}_n^\top$. Then,

$$\begin{aligned} U(U^\top S^{-1}U)^{-1}U^\top &= U(U^\top (aI_n - b\mathbf{1}_n \mathbf{1}_n^\top)U)^{-1}U^\top \\ &= U(aW - bvv^\top)^{-1}U^\top, \end{aligned} \tag{9}$$

where $W = U^\top U$ and $v = U^\top \mathbf{1}_n$. By the Sherman-Morrison-Woodbury identity, we have that

$$\begin{aligned} (aW - bvv^\top)^{-1} &= a^{-1}W^{-1} - \frac{a^{-1}W^{-1}(-bv)v^\top a^{-1}W^{-1}}{1 + v^\top a^{-1}W^{-1}(-bv)} \\ &= \frac{1}{a}W^{-1} + \frac{b}{a(a - b\varsigma)}W^{-1}vv^\top W^{-1}, \end{aligned}$$

where $\varsigma = v^\top W^{-1}v$ (it is easy to show that $a - b\varsigma > 0$). Substituting this back into (9) gives us

$$U(U^\top S^{-1}U)^{-1}U^\top = \frac{1}{a}Z + \frac{b}{a(a - b\varsigma)}zz^\top, \quad (10)$$

where $Z = U(U^\top U)^{-1}U^\top$, $z = Z\mathbf{1}_n$, and $\varsigma = \mathbf{1}_n^\top Z\mathbf{1}_n$. From the block-diagonal structure of U , the matrix Z is also block-diagonal with blocks

$$U_i(U_i^\top U_i)^{-1}U_i^\top = I_{p_i+q_i} - \frac{1}{p_i + q_i} \begin{pmatrix} \mathbf{1}_{p_i}\mathbf{1}_{p_i}^\top & -\mathbf{1}_{p_i}\mathbf{1}_{q_i}^\top \\ -\mathbf{1}_{q_i}\mathbf{1}_{p_i}^\top & \mathbf{1}_{q_i}\mathbf{1}_{q_i}^\top \end{pmatrix}, \quad \text{for } i = 1, \dots, r,$$

where we have computed $(U_i^\top U_i)^{-1}$ using Lemma 4.2. Next, one shows that $Z = I_n - Y$, $z = \mathbf{1}_n - y$, and $\varsigma = n - \gamma$. Finally, substituting

$$S^{-1} = aI_n - b\mathbf{1}_n\mathbf{1}_n^\top, \quad a = \frac{1}{\alpha}, \quad b = \frac{\ell}{\alpha(\alpha + \ell n)}, \quad (11)$$

and equation (10) into Proposition 4.8 gives the desired result. \square

Corollary 4.10. *For $1 \leq i, j \leq r$, the constants c_{ij} in the blocks $N[i, j]$ in Corollary 4.5 are:*

$$\begin{aligned} c_{ii} &= \frac{\ell}{\alpha(\alpha + \ell\gamma)(p_i + q_i)} \left(\frac{\alpha}{\ell} + \gamma - \frac{(p_i - q_i)^2}{p_i + q_i} \right), \\ c_{ij} &= \frac{-\ell}{\alpha(\alpha + \ell\gamma)} \left(\frac{p_i - q_i}{p_i + q_i} \right) \left(\frac{p_j - q_j}{p_j + q_j} \right), \quad j \neq i. \end{aligned}$$

Finally, we write down an explicit formula for $\|N\|_\infty$ and verify that $\|N\|_\infty \leq \|S^{-1}\|_\infty$.

Corollary 4.11. *If $r = 0$, then $\|N\|_\infty = 0$. If $r \geq 1$, let $d = \sum_{i=1}^r |p_i - q_i|$. Then*

$$\|N\|_\infty = \frac{1}{\alpha} + \frac{\ell}{\alpha(\alpha + \ell\gamma)} \max_{1 \leq i \leq r} \frac{|p_i - q_i|(d - 2|p_i - q_i|)}{p_i + q_i}.$$

Proof. Recall that if $r = 0$ (i.e. no component of G is bipartite), then $N = 0$ by Corollary 4.5. Now if $r \geq 1$, we may assume that $p_i \geq q_i$ for all i after a relabeling of vertices. Observe that $c_{ii} > 0$ and $c_{ij} \leq 0$ for all i and $j \neq i$. Indeed, this follows from $\alpha > 0$, $\ell > 0$, and

$$\gamma = \sum_{i=1}^r \frac{(p_j - q_j)^2}{p_j + q_j} \geq 0, \quad \gamma - \frac{(p_i - q_i)^2}{p_i + q_i} = \sum_{j \neq i} \frac{(p_j - q_j)^2}{p_j + q_j} \geq 0.$$

Consequently, the 1-norm of rows in the i -th block of N is

$$(p_i + q_i)c_{ii} - \sum_{j \neq i} (p_j + q_j)c_{ij} = \frac{1}{\alpha} + \frac{\ell(p_i - q_i)(d - 2p_i + 2q_i)}{\alpha(\alpha + \ell\gamma)(p_i + q_i)}, \quad (12)$$

and $\|N\|_\infty$ is the maximum of these 1-norms. \square

Corollary 4.12. *For all signless Laplacians P , we have*

$$N = \left\| \lim_{t \rightarrow \infty} (S + tP)^{-1} \right\|_{\infty} \leq \|S^{-1}\|_{\infty} = \frac{1}{\alpha} + \frac{\ell(n-2)}{\alpha(\alpha + \ell n)},$$

with equality if and only if P is the zero matrix.

Proof. Let $N = \lim_{t \rightarrow \infty} (S + tP)^{-1}$. If $r = 0$ then $\|N\|_{\infty} = 0$, so suppose that $r \geq 1$. As before, assume that $p_i \geq q_i$ for all $i = 1, \dots, r$. It suffices to show that the 1-norms of the rows of N , as computed in (12), are at most $\|S^{-1}\|_{\infty}$, with equality achieved only at $J = S$. The inequality is trivial if $p_i = q_i$ so we may assume $p_i - q_i \geq 1$. We outline the proof and leave the details to the reader. The key is to show that

$$(\alpha + \ell n)(d - 2p_i + 2q_i) \left(\frac{p_i - q_i}{p_i + q_i} \right) \leq (\alpha + \ell n)(d - 2) \leq (\alpha + \ell \gamma)(n - 2).$$

The latter inequality is equivalent to

$$0 \leq (n - d)(\alpha + 2\ell - \ell n) + 2\ell(d - \gamma) + n\ell(n - 2d + \gamma).$$

The first summand is nonnegative because S is diagonally dominant, while the last summand satisfies

$$n - 2d + \gamma = s + \sum_{j=1}^r \frac{4q_j^2}{p_j + q_j} \geq 0.$$

Finally, if equality $\|N\|_{\infty} = \|S^{-1}\|_{\infty}$ is achieved, then $s = 0$ and $q_j = 0$ for all j , so $P = 0$. \square

5 Analysis of $\|J^{-1}\|_{\infty}$ in a Neighborhood of S

The arguments in Sections 3 and 4 show that for $J \geq S := \alpha I_n + \ell \mathbf{1}_n \mathbf{1}_n^{\top}$, the maximum of $\|J^{-1}\|_{\infty}$ is attained at $J = S$. To prove that S is the unique maximizer, we will show that $\|J^{-1}\|_{\infty}$ is strictly decreasing near S . Let $P \geq 0$ be a nonzero symmetric diagonally dominant matrix, and consider the function

$$f(t) = \|(S + tP)^{-1}\|_{\infty}, \quad t \geq 0.$$

In our proof, we study the linear part $S^{-1} - tS^{-1}PS^{-1}$ of the Neumann series for $(S + tP)^{-1}$. Let us define $g(t) = \|S^{-1} - tS^{-1}PS^{-1}\|_{\infty}$ and $h(t) = f(t) - g(t)$. Our main result in this section is the following.

Proposition 5.1. *The function $f(t)$ is differentiable at $t = 0$ and $f'(0) < 0$.*

Proof. Since $f(t) = g(t) + h(t)$, the result follows from Propositions 5.2 and 5.3. \square

Proposition 5.2. *The function $h(t)$ is differentiable at $t = 0$ and $h'(0) = 0$.*

Proof. For sufficiently small $t > 0$, by the Neumann series for $(I + tS^{-1/2}PS^{-1/2})^{-1}$ we can write

$$(S + tP)^{-1} = S^{-1/2}(I + tS^{-1/2}PS^{-1/2})^{-1}S^{-1/2} = S^{-1/2} \left(\sum_{k=0}^{\infty} (-t)^k (S^{-1/2}PS^{-1/2})^k \right) S^{-1/2}.$$

By the reverse triangle inequality and submultiplicativity of $\|\cdot\|_{\infty}$, we have

$$\begin{aligned} |h(t)| &\leq \|(S + tP)^{-1} - S^{-1} + tS^{-1}PS^{-1}\|_{\infty} \\ &= \left\| S^{-1/2} \left(\sum_{k=2}^{\infty} (-t)^k (S^{-1/2}PS^{-1/2})^k \right) S^{-1/2} \right\|_{\infty} \\ &= t^2 \left\| S^{-1/2} (S^{-1/2}PS^{-1/2})^2 S^{-1/2} (S + tP)^{-1} \right\|_{\infty} \\ &\leq 2t^2 \|S^{-1}PS^{-1}P\|_{\infty} \|S^{-1}\|_{\infty}, \end{aligned}$$

where the last inequality holds for sufficiently small $t > 0$ since by continuity $\|(S + tP)^{-1}\|_\infty \leq 2\|S^{-1}\|_\infty$ for small t . It follows that $h'(0) = \lim_{t \rightarrow 0} (h(t) - h(0))/t = 0$. \square

Proposition 5.3. *The function $g(t)$ is differentiable at $t = 0$ and $g'(0) < 0$.*

Proof. Set $Q = S^{-1}PS^{-1}$. Note that for sufficiently small $t > 0$, the entries of $S^{-1} - tQ$ have the same sign as the corresponding entries of S^{-1} . Since $(S^{-1})_{ii} > 0$ and $(S^{-1})_{ij} < 0$ for $i \neq j$, we can write

$$\begin{aligned} g(t) &= \|S^{-1} - tQ\|_\infty = \max_i \sum_j |(S^{-1})_{ij} - tQ_{ij}| \\ &= \max_i [(S^{-1})_{ii} - tQ_{ii} + \sum_{j \neq i} (tQ_{ij} - (S^{-1})_{ij})] \\ &= \max_i [-(Q_{ii} - \sum_{j \neq i} Q_{ij})t + \|S^{-1}\|_\infty] \\ &= -\xi t + \|S^{-1}\|_\infty. \end{aligned}$$

where $\xi = \min_i (Q_{ii} - \sum_{j \neq i} Q_{ij}) > 0$ by Proposition 5.4 below. Thus, $g'(0) = -\xi < 0$ as required. \square

Proposition 5.4. *Let $Q = S^{-1}PS^{-1}$. Then $Q_{ii} - \sum_{j \neq i} Q_{ij} > 0$ for all i .*

Proof. For simplicity, let us write $S^{-1} = aI_n - b\mathbf{1}_n\mathbf{1}_n^\top$. Then,

$$Q = (aI - b\mathbf{1}\mathbf{1}^\top)P(aI - b\mathbf{1}\mathbf{1}^\top) = a^2P - abp\mathbf{1}^\top - ab\mathbf{1}p^\top + b^2\pi\mathbf{1}^\top$$

where $p = P\mathbf{1} = (p_1, \dots, p_n)$ and $\pi = \mathbf{1}^\top P\mathbf{1}$. It is straightforward to check that

$$Q_{ii} - \sum_{j \neq i} Q_{ij} = a^2(2P_{ii} - p_i) + abp_i(n - 4) + b\pi(a + 2b - bn).$$

From equation (11), we get $a/b = \alpha/\ell + n$. Substituting this relation and rearranging gives us

$$\begin{aligned} Q_{ii} - \sum_{j \neq i} Q_{ij} &= b^2 \left[\left(\frac{\alpha}{\ell} + n - 2 \right) \left(\frac{\alpha}{\ell} + 4 \right) + 4 \right] \Delta_i(P) + \\ &\quad b^2 \left[2 \left(\frac{\alpha}{\ell} + n - 1 \right) (n - 3) + \frac{\alpha}{\ell} \right] P_{ii} + b^2 \left(\frac{\alpha}{\ell} + 2 \right) \sum_{j, k \neq i} P_{jk}. \end{aligned}$$

Because S is diagonally dominant, we have $\alpha/\ell \geq n - 2 > 0$. It is not difficult to deduce that if $P \neq 0$, then the above expression is always positive, as required. \square

6 Proof of Theorem 1.1

Theorem 1.1 is a special case of the following theorem when $\alpha = (n - 2)\ell$.

Theorem 6.1. *Let $n \geq 3$ and suppose $S = \alpha I_n + \ell \mathbf{1}_n \mathbf{1}_n^\top$ is diagonally dominant with $\alpha, \ell > 0$. For all $n \times n$ symmetric diagonally dominant matrices $J \geq S$, we have*

$$\|J^{-1}\|_\infty \leq \|S^{-1}\|_\infty = \frac{\alpha + 2\ell(n - 1)}{\alpha(\alpha + \ell n)}.$$

Furthermore, equality is achieved if and only if $J = S$.

Proof. Recall from Section 3 that \mathcal{D}_m is the set of symmetric diagonally dominant matrices J with $\ell \leq J_{ij} \leq m$ and $\Delta_i(S) \leq \Delta_i(J) \leq \Delta_i(S) + 2(m - \ell)$ for all $i \neq j$. Recall also that $S + (m - \ell)P \in \mathcal{D}_m$ is a corner matrix if P is a signless Laplacian. Let \mathcal{T}_m be the set of matrices $J \in \mathcal{D}_m$ maximizing $\|J^{-1}\|_\infty$. We claim that for sufficiently large $m > \ell$, we have $\mathcal{T}_m = \{S\}$. Indeed, from Corollary 4.12 and for large m :

$$\|J^{-1}\|_\infty < \|S^{-1}\|_\infty, \quad \text{for all corner matrices } J \in \mathcal{D}_m \setminus \{S\}.$$

Thus, by Proposition 3.1, every $J \in \mathcal{T}_m$ must be path-connected to the corner matrix S . Since Proposition 5.1 implies that S is an isolated point in \mathcal{T}_m , we must have $\mathcal{T}_m = \{S\}$ as claimed.

Finally, suppose $J^* \geq S$ is a symmetric diagonally dominant matrix with $\|(J^*)^{-1}\|_\infty \geq \|S^{-1}\|_\infty$. We will show that $J^* = S$, which proves Theorem 1.1. We assume that m is sufficiently large with $J^* \in \mathcal{D}_m$ and $\mathcal{T}_m = \{S\}$. Then S is the unique maximizer of $\|J^{-1}\|_\infty$ for $J \in \mathcal{D}_m$ so that $J^* = S$, as desired. \square

7 Extensions of Hadamard's Inequality

Our arguments for proving Theorem 1.5 are inspired by block LU factorization ideas in [3]. For $1 \leq i \leq n$, let $J_{(i)}$ be the lower right $(n-i+1) \times (n-i+1)$ block of J , so $J_{(1)} = J$ and $J_{(n)} = (J_{nn})$. Also, for $1 \leq i \leq n-1$, let $b_{(i)} \in \mathbb{R}^{n-i}$ be the column vector such that

$$J_{(i)} = \begin{pmatrix} J_{ii} & b_{(i)}^\top \\ b_{(i)} & J_{(i+1)} \end{pmatrix}.$$

Then our block decomposition takes the form, for $1 \leq i \leq n-1$,

$$J_{(i)} = \begin{pmatrix} 1 & U_{(i)} \\ 0 & I_{n-i} \end{pmatrix} \begin{pmatrix} s_i & 0 \\ b_{(i)} & J_{(i+1)} \end{pmatrix}$$

with

$$s_i = J_{ii} \left(1 - \frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}} \right) \quad \text{and} \quad U_{(i)} = b_{(i)}^\top J_{(i+1)}^{-1}.$$

Notice that $\det(J) = J_{nn} \prod_{i=1}^{n-1} s_i$, or equivalently,

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} = \prod_{i=1}^{n-1} \frac{s_i}{J_{ii}} = \prod_{i=1}^{n-1} \left(1 - \frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}} \right). \quad (13)$$

It remains to bound each factor s_{ii}/J_{ii} . We first establish the following results.

Lemma 7.1. *Let J be a symmetric diagonally balanced $n \times n$ matrix with $0 < \ell \leq J_{ij} \leq m$ for $i \neq j$. Then $\ell S \preceq J \preceq mS$, and the eigenvalues $\lambda_1 \leq \dots \leq \lambda_n$ of J satisfy*

$$(n-2)\ell \leq \lambda_i \leq (n-2)m \quad \text{for } 1 \leq i \leq n-1 \quad \text{and} \quad 2(n-1)\ell \leq \lambda_n \leq 2(n-1)m.$$

Moreover, if J is diagonally dominant, then the lower bounds still hold.

Proof. We first show that if $P \geq 0$ is a symmetric diagonally dominant matrix, then $P \succeq 0$. For any $x \in \mathbb{R}^n$,

$$x^\top P x = \sum_{i=1}^n P_{ii} x_i^2 + 2 \sum_{i < j} P_{ij} x_i x_j \geq \sum_{i=1}^n \left(\sum_{j \neq i} P_{ij} \right) x_i^2 + 2 \sum_{i < j} P_{ij} x_i x_j = \sum_{i < j} P_{ij} (x_i + x_j)^2 \geq 0.$$

Since the matrices $P = J - \ell S$ and $Q = mS - J$ are symmetric and diagonally balanced with nonnegative entries, it follows that $P, Q \succeq 0$ by the discussion above, which means $\ell S \preceq J \preceq mS$. The eigenvalues of S are $\{n-2, \dots, n-2, 2(n-1)\}$, so the result follows by an application of [5, Corollary 7.7.4]. If J is diagonally dominant, then $\ell S \preceq J$, and hence the lower bounds, still holds. \square

Lemma 7.2. *Let J be a symmetric diagonally balanced $n \times n$ matrix with $0 < \ell \leq J_{ij} \leq m$ for $i \neq j$. For each $1 \leq i \leq n$, let $J_{(i)}$ be the lower right $(n-i+1) \times (n-i+1)$ block of J as defined above, and suppose the eigenvalues of $J_{(i)}$ are $\lambda_1 \leq \dots \leq \lambda_{n-i+1}$. Then*

$$(n-2)\ell \leq \lambda_j \leq (n-2)m \quad \text{for } 1 \leq j \leq n-i \quad \text{and} \quad (2n-i-1)\ell \leq \lambda_{n-i+1} \leq (2n-i-1)m.$$

Moreover, if J is diagonally dominant, then the lower bounds still hold.

Proof. Write $J_{(i)} = H + D$, where H is the $(n-i+1) \times (n-i+1)$ diagonally balanced matrix and D is diagonal with nonnegative entries. Note that $(i-1)\ell I \preceq D \preceq (i-1)mI$, so $(i-1)\ell I + H \preceq J_{(i)} \preceq (i-1)mI + H$. Thus by [5, Corollary 7.7.4] and by applying Lemma 7.1 to H , we get, for $1 \leq j \leq n-i$,

$$(n-2)\ell = (n-i-1)\ell + (i-1)\ell \leq \lambda_j \leq (n-i-1)m + (i-1)m = (n-2)m,$$

and for $j = n - i + 1$,

$$(2n - i - 1)\ell = 2(n - i)\ell + (i - 1)\ell \leq \lambda_{n-i+1} \leq 2(n - i)m + (i - 1)m = (2n - i - 1)m.$$

If J is diagonally dominant, then $(i - 1)\ell I + H \preceq J_{(i)}$ and hence the lower bounds still hold. \square

Proof of Theorem 1.5: For part (a), suppose J is diagonally dominant. For each $1 \leq i \leq n - 1$ we have $J_{ii} \geq \sum_{j \neq i} J_{ij} \geq b_{(i)}^\top \mathbf{1}_{n-i}$, and by Lemma 7.2, the maximum eigenvalue of $J_{(i+1)}^{-1}$ is at most $\frac{1}{(n-2)\ell}$. Thus,

$$\frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}} \leq \frac{1}{(n-2)\ell} \frac{b_{(i)}^\top b_{(i)}}{J_{ii}} \leq \frac{1}{(n-2)\ell} \frac{b_{(i)}^\top b_{(i)}}{b_{(i)}^\top \mathbf{1}} \leq \frac{\sqrt{(n-i+1)m}}{(n-2)\ell} \frac{\sqrt{b_{(i)}^\top b_{(i)}}}{b_{(i)}^\top \mathbf{1}}.$$

Since each entry of $b_{(i)}$ is bounded by ℓ and m , the reverse Cauchy-Schwarz inequality [15, Ch. 5] gives us

$$\frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}} \leq \frac{\sqrt{(n-i+1)m}}{(n-2)\ell} \frac{\ell + m}{2\sqrt{\ell m(n-i+1)}} = \frac{1}{2(n-2)} \sqrt{\frac{m}{\ell}} \left(1 + \frac{m}{\ell}\right).$$

Substituting this inequality into (13) gives us the desired bound for part (a).

For part (b), suppose J is diagonally balanced, so $J_{ii} \leq (n - 1)m$ for each $1 \leq i \leq n - 1$. By Lemma 7.2 we know that the minimum eigenvalue of $J_{(i+1)}^{-1}$ is at least $\frac{1}{(2n-i-2)m}$, so

$$\frac{b_{(i)}^\top J_{(i+1)}^{-1} b_{(i)}}{J_{ii}} \geq \frac{1}{(2n-i-2)m} \frac{b_{(i)}^\top b_{(i)}}{J_{ii}} \geq \frac{1}{2(n-1)m} \frac{(n-i+1)\ell^2}{(n-1)m} = \frac{(n-i+1)\ell^2}{2(n-1)^2 m^2}.$$

Substituting this into (13) and using the inequality $1 + x \leq \exp(x)$ gives us the desired bound:

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} \leq \prod_{i=1}^{n-1} \exp\left(-\frac{(n-i+1)\ell^2}{2(n-1)^2 m^2}\right) = \exp\left(-\frac{(n+2)\ell^2}{4(n-1)m^2}\right) \leq \exp\left(-\frac{\ell^2}{4m^2}\right). \quad \square$$

We close this section with several examples.

Example 7.3. The matrix $S = (n - 2)I_n + \mathbf{1}_n \mathbf{1}_n^\top$ has eigenvalues $\{n - 2, \dots, n - 2, 2(n - 1)\}$, so

$$\frac{\det(S)}{\prod_{i=1}^n S_{ii}} = \frac{2(n-2)^{n-1}(n-1)}{(n-1)^n} = 2 \left(1 - \frac{1}{n-1}\right)^{n-1} \rightarrow \frac{2}{e} \quad \text{as } n \rightarrow \infty. \quad \square$$

Example 7.4. When J is strictly diagonally dominant, the ratio $\det(J)/\prod_{i=1}^n J_{ii}$ can be arbitrarily close to 1. For instance, consider $J = \alpha I_n + \mathbf{1}_n \mathbf{1}_n^\top$ with $\alpha \geq n - 2$, which has eigenvalues $\{(n + \alpha), \alpha, \dots, \alpha\}$ so

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} = \frac{(n + \alpha)\alpha^{n-1}}{(\alpha + 1)^n} \rightarrow 1 \quad \text{as } \alpha \rightarrow \infty. \quad \square$$

Example 7.5. The following example demonstrates that we require an upper bound on the entries of J in Theorem 1.5 (a). Let $n = 2k$ for some $k \in \mathbb{N}$, and consider the matrix J in the following block form:

$$J = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \quad A = (km + k\ell - 2\ell)I_k + \ell \mathbf{1}_k \mathbf{1}_k^\top, \quad B = m \mathbf{1}_k \mathbf{1}_k^\top.$$

By the determinant block formula (since A and B commute), we have

$$\begin{aligned} \det(J) &= \det(A^2 - B^2) \\ &= \det\left[(km + k\ell - 2\ell)^2 I_k + (2k\ell m + 3k\ell^2 - 4\ell^2 - km^2) \mathbf{1}_k \mathbf{1}_k^\top\right] \\ &= 4\ell(k-1)(km + k\ell - \ell) \cdot (km + k\ell - 2\ell)^{2k-2}, \end{aligned}$$

where the last equality is obtained by considering the eigenvalues of $A^2 - B^2$. Then

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} = \frac{4\ell(k-1)(km + k\ell - \ell) \cdot (km + k\ell - 2\ell)^{2k-2}}{(km + k\ell - \ell)^{2k}} \rightarrow \frac{4\ell}{\ell + m} \exp\left(-\frac{2\ell}{\ell + m}\right) \text{ as } k \rightarrow \infty.$$

Note that the last quantity above tends to 0 as $m/\ell \rightarrow \infty$. \square

8 Open Problems

As an analogue to Theorem 6.1, we also conjecture a tight lower bound for symmetric diagonally dominant matrices $J > 0$ whose off-diagonal entries and diagonal dominances are bounded above. Observe that when $J_{ij} \leq m$ and $\Delta_i(J) \leq \delta$ for all $i \neq j$, then $J \leq (m(n-2) + \delta)I_n + m\mathbf{1}_n\mathbf{1}_n^\top$.

Conjecture 8.1. *Let $n \geq 3$ and let $S(\alpha, m) = \alpha I_n + m\mathbf{1}_n\mathbf{1}_n^\top$. For all $n \times n$ symmetric diagonally dominant matrices $0 < J \leq S(\alpha, m)$, we have*

$$\|J^{-1}\|_\infty \geq \|S(\alpha, m)^{-1}\|_\infty = \frac{\alpha + 2m(n-1)}{\alpha(\alpha + mn)}.$$

Moreover, equality is achieved if and only if $J = S(\alpha, m)$.

In working towards the proof of this conjecture, the following problem may be useful.

Problem 8.2. *Given a signless Laplacian P of a graph G , give an exact combinatorial formula for the entries of $(S + tP)^{-1}$ for any $t > 0$. More precisely, since each entry of $(S + tP)^{-1}$ is a rational function of t , derive a formula for the coefficients of this rational function in terms of the combinatorics of the graph G .*

We also conjecture that a dependence on the largest entry can be removed in Theorem 1.5 (b).

Conjecture 8.3. *For a positive, diagonally balanced symmetric J , we have the bound:*

$$\frac{\det(J)}{\prod_{i=1}^n J_{ii}} \leq \frac{\det(S)}{(n-1)^n} = 2 \left(1 - \frac{1}{n-1}\right)^{n-1} \rightarrow \frac{2}{e}.$$

References

- [1] S. Chatterjee, P. Diaconis, A. Sly. *Random graphs with a given degree sequence*. Annals of Applied Probability, **21** (2011), 1400–1435.
- [2] D. Cvetković and S. Simić. *Towards a spectral theory of graphs based on the signless Laplacian, III*. Applicable Analysis and Discrete Mathematics, 4(1):156–166, 2010.
- [3] J.W. Demmel, N.J. Higham, and R. Schreiber. *Block LU factorization*. Research Institute of Advanced Computer Science, 1992.
- [4] C. Hillar and A. Wibisono. *Maximum entropy distribution on graphs with a given degree sequence*. In preparation.
- [5] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 1990.
- [6] R.A. Horn and C.R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, 1991.
- [7] C.R. Johnson and C. Hillar. *Eigenvalues of words in two positive definite letters*. SIAM Journal on Matrix Analysis and Applications, **23** (2002), 916–928.

- [8] S. Lang. *Real and Functional Analysis*. Springer, 1993.
- [9] W. Li. *The infinity norm bound for the inverse of nonsingular diagonal dominant matrices*. Applied Mathematics Letters, **21** (2008), 258–263.
- [10] J.R. Munkres. *Topology (2nd Edition)*. Prentice Hall, 2000.
- [11] R. Sanyal, B. Sturmfels, C. Vinzant. *The entropic discriminant*. <http://arxiv.org/abs/1108.2925>, 2011.
- [12] P.N. Shivakumar, J.J. Williams, Q. Ye, and C.A. Marinov. *On two-sided bounds related to weakly diagonally dominant M-matrices with application to digital circuit dynamics*. SIAM Journal on Matrix Analysis and Applications, **17** (1996) 298.
- [13] D.A. Spielman and S.H. Teng. *Nearly-linear time algorithms for graph partitioning, graph sparsification, and solving linear systems*. Proceedings of the thirty-sixth annual ACM symposium on Theory of computing (STOC '04), (2004), 81–90.
- [14] D.A. Spielman and S.H. Teng. *Nearly-linear time algorithms for preconditioning and solving symmetric, diagonally dominant linear systems*. <http://arxiv.org/abs/0808.4134>, 2006.
- [15] J.M. Steele. *The Cauchy-Schwarz Master Class*. Cambridge University Press, 2004.
- [16] J.M. Varah. *A lower bound for the smallest singular value of a matrix*. Linear Algebra and its Applications, **11** (1975), 3–5.
- [17] R.S. Varga. *On diagonal dominance arguments for bounding $\|A^{-1}\|_{\infty}$* . Linear Algebra and its Applications, **14** (1976), 211–217.